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Convergence and Performance of Synchronous and  
Asynchronous Parallel and Conventional Iterative  
Methods

**FINAL TECHNICAL REPORT**

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## Table of Contents

Research Progress and Activity Report .....	1
Asynchronous Algorithms .....	5
The Determination of Nonnegative Solutions to Linear Systems of Differential Equations by Finite Differences Methods .....	12
References .....	14
P.I.'s Vitae .....	17
Recent Technical Reports .....	26+



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## 1 RESEARCH PROGRESS AND ACTIVITIES REPORT

During the last eighteen months of the grant our work concentrated on two main research topics:

(i) **“Parallel and Sequential Iterative Methods for Linear and Non-linear Systems”.** Much of the work on this topic concentrated on the convergence and rate of convergence of parallel asynchronous methods for solving linear systems arising, on the one hand, from the numerical solution to partial differential equations and, on the other hand, from least squares solution to rectangular systems which arise in application such as image reconstruction from incomplete tomographical data. The mathematics behind the analysis of these two applications of the asynchronous parallel methods is quite different.

Recently we have been able to extend our convergence results to asynchronous methods for solving nonlinear systems. One application now consists of tomographic reconstruction from incomplete data where the image is constrained to lie in a bounded convex set such as an  $n$  dimensional box.

(ii) **“Reachability Problems for Dynamical Systems”.** Here we concentrated on developing numerical methods to test whether the trajectory of a linear differential system emanating from a given initial state becomes from some point onwards nonnegative. We particularly characterized when such initial states are *symbiosis points*, meaning that from some point onwards all populations become *nondecreasing*.

As a by-product of the work on these topics we also had to solve various theoretical problems which can be described under the heading:

(iii) **“Problems in Nonnegative Matrices and their Applications”.**

We shall describe the main results which were obtained on these topics in the next section of this report. We strongly believe that an examination of the results which were achieved over the life of the grant shows that many of the goals which were suggested in the initial 1987 proposal and in the annual

and progress reports which have been submitted since, have been realized. Quite a few of the questions that have been raised have been answered, but not always with the solution that was conjectured.

Since the beginning of the work on the proposal, 15 papers which summarize our results on the above three topics have been submitted for publication. Their titles are as follows:

**On parallel and sequential iterative methods**

- 1) (with P. J. Kavanagh) "Consistency and convergence of the parallel multisplitting method for singular M-matrices," SIAM J. Matrix Analysis Appl., 10(1989), pp. 210-218.
- 2) (with L. Elsner and I. Koltracht) "On the convergence of asynchronous paracontractions with application to tomographic reconstruction from incomplete data," Lin. Alg. Appl., 130(1990), pp.65-82.
- 3) (with A. Hadjidimos) "Convergence domains of the SSOR method for generalized consistently ordered matrices", J. Comp. Appl. Math., 33(1990), pp. 35-52.
- 4) (with M. Hanke) "Preconditioning and splittings for rectangular systems", Numer. Math., 57(1990), pp.85-95.
- 5) (with E. Elsner and B. Vemmer) "The effect of the number of processors on the convergence of the parallel block Jacobi method", Lin. Alg. Appl., 154-156(1991), pp.311-330.
- 6) (with M. Hanke and W. Niethammer) "On the SOR method for symmetric positive definite systems", Lin. Alg. Appl., 154-156(1991), pp.457-472.
- 7) (with L. Elsner) "Monotonic sequences and rates of convergence of asynchronous iterative methods", submitted to Numer. Math.
- 8) (with L. Elsner and I. Koltracht) "Convergence of sequential and asynchronous nonlinear paracontractions", submitted to Numer. Math.

**On the reachability problem**

- 9) (with R. J. Stern) " Discrete approximations to reachability cones of linear differential equations," Lin. Alg. Appl., 120(1989), pp. 65-79.

- 10) (with R. J. Stern and M. Tsatsomeros) "The reachability cones of essentially nonnegative matrices", *Lin. Multilin. Alg.*, 28(1991), pp.213-224.
- 11) (with M. Tsatsomeros) "Symbiosis points for linear differential systems", *Lin. Multilin. Alg.*, 30(1991), pp.49-59.

On nonnegative matrices and applications

- 12) (with R. E. Hartwig and N. J. Rose) "An algebraic-analytic approach to nonnegative basis," *Lin. Alg. Appl.*, 133(1990), pp.77-88.
- 13) (with I. Koltracht) "On the inverse M-matrix problem for real symmetric positive definite Toeplitz matrices", *SIAM J. Matrix Anal. Appl.*, 12(1991), pp.310-320.
- 14) (with H. J. Werner) "Nonnegative group inverses", *Lin. Alg. Appl.*, 151(1991), pp.85-96.
- 15) (with H. Schneider) "Principal components of minus M-matrices", submitted to *Lin. Multilin. Alg.*

During the first half of the period in which this grant has been in effect we have also co-authored a book in connection with the second research topic listed above. The title of the book is "Nonnegative Matrices in Dynamic Systems". Its other authors are A. Berman and R. J. Stern and it was published in the Series in Pure and Applied Mathematics, Wiley Interscience, New York, 1989.

Since the start of the grant we have presented results from our work and acknowledged the support of the Air Force in conferences and colloquia as follow: "Special Session on Modern Trends in Matrix Theory and its Applications", AMS Annual Meeting, January, 1988; "Oberwolfach Meeting on Numerical Algebra and Parallel Computations", Oberwolfach, February 1988; "Workshop on Iterative Solutions to Singular Systems", Univ. of Karlsruhe, Karlsruhe, West Germany, March 1988; "SIAM 3rd Conference on Applied Linear Algebra", Madison, May 1988; "International Symposium on Computational Applied Math.", Leuven, Belgium, July 1988; "NATO Advanced Study Institute on Numerical Linear Algebra, Digital Signal Processing and Parallel Algorithms", Leuven, Belgium, August 1988; "Conference on Iterative Methods for Large Linear Systems (dedicated to David

M. Young)", University of Texas, Austin, October 1988; "A Conference on Approximation Theory and Numerical Linear Algebra (dedicated to R. S. Varga)", Kent State University, Kent, Ohio, March 1989; Northern Illinois University Conference on "Linear Algebra, Numerical Linear Algebra and Applications", DeKalb, Illinois, April 1989; "The Householder Symposium XI on Numerical Linear Algebra", Tylösand, Sweden, June 1990; Meeting on "Numerical Linear Algebra", Oberwolfach, Germany, April 1991; "Haifa VII Matrix Theory Conference", Haifa, Israel, June 1991.

The referencing within the report is as follows. Papers cited, but not co-authored by the P.I., are referenced by numbers in the text and the key is given after section 3. References to papers co-authored by the P.I. are numbered by [Nxx], where xx refers to the paper number in the P.I.'s *vitae* which is attached at the end of this report.

## 2 ASYNCHRONOUS ALGORITHMS FOR LARGE LINEAR SYSTEMS

In this section we describe our research concerning the convergence and the acceleration of convergence of a certain model of a *parallel chaotic* (also known as *asynchronized*) iteration scheme. Some of the problems we encountered come from the fact that we tried to apply the same synchronized model to linear systems whose coefficient matrices arises in different applications and also to nonlinear systems. This means that for each application we had to find the inherent mathematical properties which make the convergence of the algorithm and its acceleration possible. Each type of system, in turn, gives rise to different problems in the actual implementation of the algorithm.

The chaotic iteration method which we have in mind has the following form: We are given

- i)  $m$  linear or nonlinear operators  $B_1, \dots, B_m$ .
- ii) A *computation cycle*, namely, a fixed time period  $T > 0$ , and a *regulated sequence integers on  $m$* , that is a sequence of integers  $\{i_j\}_{j=1}^{\infty}$  with  $1 \leq i_j \leq m$  and such that

$$\{1, 2, \dots, m\} \subseteq \{i_j, \dots, i_{j+T-1}\}, \forall j \geq 1. \quad (2.1)$$

- iii)  $m$  nonnegative diagonal matrices  $E_l$ ,  $l = 1, \dots, m$ , whose sum is the identity matrix. (They are sometimes called *weighting* or *masking* matrices.)

- iv) A parallel machine with  $k$  processors and a host node.

We perform the iteration:

$$x^{(j+r_j)} = (I - E_{i_j})x^{(j+r_j-1)} + E_{i_j}B_{i_j}x^{(j)}, \quad j = 1, 2, \dots \quad (2.2)$$

Our model works as follows: At time  $j$  a processor, call it for now the **subject** processor, which has just completed a previous task is assigned the task specified by  $i_j$ , namely, by the operators  $E_{i_j}$  and  $B_{i_j}$ . This means that it begins to calculate the vector  $u = E_{i_j}B_{i_j}x^{(j)}$ . Note that only the entries of  $u$  corresponding to the nonzero diagonal entries in  $E_{i_j}$  need be computed. The number  $r_j - 1$ ,  $r_j \leq T$ , then represents the number of

similar tasks completed by other processors before the subject processor completes its present computation. When this computation is done, the sum  $x^{(j+r)} = u + (I - E_{i_j})x^{(j+r-1)}$  is formed by the host processor and the subject processor is then assigned task  $i_{j+r}$ . The existence of a computation cycle  $T > 0$  as given by (2.1) means that there exists a time period  $T$  such that in every  $T$  successive iteration, the global approximation to the solution is corrected at least once by each of the operators  $B_1, \dots, B_m$ .

We have proved the convergence of (2.2) when the chaotic process is used to solve iteratively two quite different types of linear systems

$$Ax = b. \quad (2.3)$$

The first type usually arises from a finite differences approximation to second order partial differential equations subject to boundary value conditions. There  $A$  is frequently a *monotone* matrix, meaning it is nonsingular with  $A^{-1} \geq 0$ . The operators  $B_l$ ,  $l = 1, \dots, m$ , are then iteration matrices induced by  $m$  *weak regular splittings* of the matrix  $A$ , that is, by  $m$  splittings

$$A = M_l - N_l, \quad (2.4)$$

satisfying  $M_l$  is invertible with

$$M_l^{-1} \geq 0 \text{ and } B_l = M_l^{-1}N_l \geq 0. \quad (2.5)$$

The second type of linear systems to which we have applied (2.2) are rectangular systems (2.3) which arise in image reconstruction from incomplete data as, for example, in well-to-well tomography used in geophysics. Previously the *cyclic Algebraic Reconstruction Technique* (ART), which itself is a generalization of the Kaczmarz projection method, has been applied to find the least squares solution of minimal norm to such systems. The cyclic ART method, which is also closely related to the *successive overrelaxation* (SOR) method (see Koltracht and Lancaster [1] and Hanke and Niethammer [2]), is a sequential method where we apply in a cyclic order the  $m$  operators  $B_l$ , each of which corresponds to an orthogonal projection onto a subspace spanned by a row or a group of rows of the coefficient matrix  $A$  with each row of the  $A$  appearing in at least one of the groups (thus overlapping is allowed). The proof that the cyclic ART-SOR can be parallelized according to (2.2) is more intricate than in the case of chaotic iteration for solving monotone system. It requires certain norm considerations and restrictions on the weighting diagonal matrices which are not necessary in the case of

chaotic iterations for monotone systems.

For both types of linear systems mentioned above we have found various proofs for the convergence of (2.2) all of which involve the embedding of the process as a sequential iteration process which takes place in higher dimensionsal space. In one of the types of embedding, we iterate sequentially in the  $kn$ -dimensional space (recall  $k$  is the number of processors and  $n$  is the dimension of the iterates in (2.2)) and produce a sequence of iteration vectors  $z^{(j)}, j = 1, 2, \dots$ . The idea now is to prove that the  $k$  subvectors of  $z^{(j)} \in R^{nk}$  each has a limit point, as  $j \rightarrow \infty$ , equal to the solution of (2.3). Although we have not used this name **formally** in any of our reports, we like to refer **informally** to this approach to the proof as the "logbook" approach. This is because for any processor, say the  $\nu$ -th, we think of the sequence of  $n$ -vectors which we can form from the  $\nu$ -th subvectors of the  $z^{(j)}$  as keeping a "logbook" of the local approximations in the  $\nu$ -th processor at each time step of the global iteration. Thus much of the time only the index of the iteration in the processor is advanced, but the actual value of the local approximation is unchanged. It only changes when the  $\nu$ -th processor updates to and downdates from the host processor.

The second type of embedding we have used to prove the convergence of (2.2) is by blowing up the process in the  $n$ -dimensional space to a sequential process in the  $nT$ -dimensional space, where  $T$  is the computation cycle. This is achieved by looking at the iteration

$$y^{(j+r)} = C_{j+r} y^{(j+r-1)} + \tilde{b},$$

where

$$C_{j+r} = \underbrace{\begin{pmatrix} (I - E_{i,j}) & 0 & \dots & 0 & E_{i,j} B_{i,j} & 0 & \dots & 0 \\ I & 0 & & & 0 & & & \\ 0 & I & & & & & & \\ & & \ddots & & 0 & & & \\ 0 & \dots & \dots & 0 & I & & \ddots & \\ & & & & & \ddots & & \\ 0 & \dots & \dots & 0 & & & & I & 0 \end{pmatrix}}_{r_j \text{ blocks}} \quad (2.6)$$

and where

$$y^{(s)} = \begin{pmatrix} x^{(s)} \\ x^{(s-1)} \\ \vdots \\ x^{(s-T+1)} \end{pmatrix} \text{ and } \bar{b} = \begin{pmatrix} b \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Above, under the appropriate assumptions on  $A$  and the  $B_l$ 's, a contraction occurs at least every time a succession of  $2T$  iterations has been applied. In otherwords, in an appropriately induced norm on  $R^{nT}$ ,  $\|C_{k+2T-1} \dots C_k\| < 1, \forall k \geq 1$ .

*One of the goals of our research was to obtain a better understanding of the meaning and the interpretation of the two embeddings. First, we can view  $r_j$  as the time elapsed (= time-lag) between two updates of the global approximation by the operator  $i_j$ . Thus if  $r_j$  is large, then the subject processor which received the global approximation  $x^{(j)}$  is, by the time it has finished computing  $u$ , using a correction based on a **relatively old approximation** to update the current global approximation in the host node. Using special nested subcones of monotonic vectors in  $R_+^{nT}$ , the cone of nonnegative vectors in the  $R^{nT}$ -dimensional space, we were able to prove in [N61] the following result on the rate of convergence of the model given in (2.2).*

**Theorem 1** *Let  $\{s_j\}_{j=1}^\infty$  and  $\{r_j\}_{j=1}^\infty$  be two sequences of time-laggs such that*

$$1 \leq s_j \leq r_j \leq T, \quad j = 1, 2, \dots \quad (2.7)$$

*If*

$$s_{j+1} \leq s_j + 1, \quad j = 1, 2, \dots, \quad (2.8)$$

*then*

$$\sup_{y \in R^{nT}} \limsup_{j \rightarrow \infty} \|\tilde{C}_j \dots \tilde{C}_1 y - \xi\|^{1/j} \leq \sup_{y \in R^{nT}} \limsup_{j \rightarrow \infty} \|C_j \dots C_1 y - \xi\|^{1/j}, \quad (2.9)$$

where

$$\tilde{C}_{j+s_j} = \underbrace{\begin{pmatrix} (I - E_{i_j}) & 0 & \dots & 0 & E_{i_j} B_{i_j} \\ I & 0 & & & 0 \\ 0 & I & & & \\ & & \ddots & & 0 \\ 0 & \dots & \dots & 0 & I \\ & & & & \ddots \\ 0 & \dots & \dots & 0 & I & 0 \end{pmatrix}}_{s_j \text{ blocks}}$$

and where  $\xi = (\bar{x}^T, \dots, \bar{x}^T)^T \in R^{nT}$  with  $\bar{x}$  being the solution to (2.3).

Let us refer to the iterative processes induced by the sequences of time-laggs  $\{s_j\}_{j=1}^\infty$  and  $\{r_j\}_{j=1}^\infty$  as the **more frequently updating process** and the **more infrequently updating process**, respectively. What the above result says is this: When  $s_{j_0+1} > s_{j_0} + 1$  for some  $j_0 \geq 1$ , then the more frequently updating process uses an older approximation to compute the  $j_0$ -th iteration than the approximation it has used in computing the immediately preceding iteration. Therefore condition (2.8) means that when the more frequently updating process never "suddenly" uses an older approximation in computing some iterate than the approximation it has used in computing the previous iterate, then the rate of convergence of the more frequently updating iteration is more favorable than the rate of convergence of the more infrequently updating. We have shown by means of examples that if condition (2.7) holds, but condition (2.8) does not, then the result of the theorem is not true.

We have carried out many numerical experiments in connection with Theorem 1 and several such are given in [N57]. There we considered the special case when:

$$s_j = k = \text{const. and } r_j = k' = \text{const.}, \forall j \geq 1$$

and with

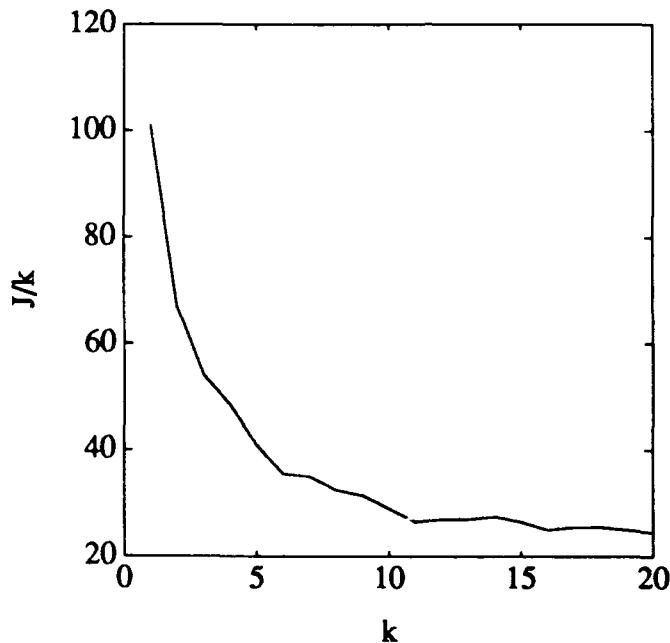
$$k \leq k'.$$

Such a delay will occur when the work among the processors is equally distributed in which case the constant delay is just the number of processors minus 1. In the following table, for a typical  $80 \times 80$  diagonally dominant matrix, we show how the increasing the number of processors (viz. increasing the delay) effects the number of iteration which are necessary to reach a given accuracy of  $10^{-6}$ . In the table  $k$  is the number of processors used by the machine and  $J = J(k)$  is the number of iterations required to achieve a prescribed accuracy to the solution. The last column is the ratio of the number of iterations to the number of processors, indicating, roughly, the number of iterations which each of the processors would have to execute in parallel if communication overheads are reasonable:

<b>k</b>	<b>J</b>	<b>J/k</b>	<b>k</b>	<b>J</b>	<b>J/k</b>
---	---	-----	---	---	-----
1	101	101.0	11	291	26.5
2	134	67.0	12	324	27.0
3	162	54.0	13	352	27.1
4	194	48.5	14	384	27.4
5	204	40.8	15	394	26.3
6	212	35.3	16	402	25.1
7	244	34.9	17	434	25.5
8	261	32.6	18	454	25.3
9	282	31.3	19	474	24.9
10	291	29.1	20	484	24.2

To give a better illustration of this table of iterates let us graph  $k$  versus

$J/k$ :



The graph clearly points to a conjecture that we have that as the number of processors increase the speed-up tends to a linear constant and no gain is achieved by increasing heavily the number of processors.

For rectangular systems of equations we have considered chaotic iterations for computing least squares solutions. As mentioned earlier, the mathematics that is needed to demonstrate that such iterations converge is quite different than for monotone systems. We attach to this report a recent paper in which we show that chaotic methods of the form (2.2) can be applied also to nonlinear systems of equations. This allows us also to consider applications to finding linear least squares solutions lying in some closed convex set which represents a nonlinear constraint on the solution. Such an application arises in computed tomography from incomplete data.

### 3 The Determination of Nonnegative Solutions to Linear Systems of Differential Equations by Finite Differences Methods

Consider the system of linear differential equations

$$\dot{x} = Ax, \quad (3.1)$$

where  $A = (a_{ij})$  is an  $n \times n$  real matrix. In many engineering, biological, and other applications the vector  $x(t)$  represents the state of a system at time  $t$  and its components frequently represent the sizes of *populations* or *species* at time  $t$ . In some applications (see Luenberger [1979]) the matrix  $A$  is essentially nonnegative, that is,  $a_{ij} \geq 0$ ,  $i \neq j$ . Such a constraint ensures that trajectories which emanate from nonnegative initial states remain nonnegative. Moreover interest centers on trajectories which eventually become and remain nonnegative or trajectories whose velocities (derivatives) become and remain nonnegative. **If the latter condition holds, then, in time, the system reaches a state from which every species will not decrease in size thereafter.** We call initial points whose trajectories and their velocities eventually become and remain nonnegative **it symbiosis points**.

From now on we shall suppose that  $A$  is essentially nonnegative. Denote by  $X_A(\mathbb{R}_+^n)$  the set of all points in  $\mathbb{R}^n$  such that the trajectories emanating from these points become and, due to the essential nonnegativity of  $A$ , remain nonnegative. In [N33] we showed that  $X_A(\mathbb{R}_+^n)$  is a convex cone which, however, need not be closed or pointed. In a sequence of papers [N31], [N33], and [N37] we gave formulas for the closure of  $X_A(\mathbb{R}_+^n)$  under various further assumptions on  $A$  such as diagonalizability, real spectrum, etc. These formulas were very difficult to apply for two reasons: (i) they were too complicated as they involved the intersections of the eigenspaces of  $A$  with various projections of the nonnegative orthant, and (ii) as only the closure of the reachability cone was determined, there were further complications in applying the formulas to determine whether a given boundary point of  $X_A(\mathbb{R}_+^n)$  is also a reachability point.

Because of the difficulties we described above we thought of the possibility of applying finite differences methods in order to determine whether a given point  $x_0 \in \mathbb{R}^n$  is a reachability point. In the simplest finite differences

schemes, the so called *Cauchy-Euler method*, we approximate the solution at times  $k = 1, 2, \dots$  from the quotients

$$\frac{\hat{x}_k - \hat{x}_{k-1}}{h} = A\hat{x}_{k-1}, \quad (3.2)$$

where  $h$  is the time-step used by the method. After some rearrangement we obtain from (3.2) the *discrete trajectory of points* emanating from  $\hat{x}_0 = x_0$  given by

$$\hat{x}_k = (I + hA)^k \hat{x}_0, \quad k = 1, 2, \dots \quad (3.3)$$

We see that discretization schemes for systems of ordinary differential equations thus resemble error analysis for iterative solutions to linear systems of equations in the sense that both procedures involve powering-up matrices. It is therefore of no surprise that underlying both are basic features and problems of the power method for determining eigenvalues and invariant subspaces.

Observe that if the time step  $h$  is small enough to make the matrix  $I + hA$  nonnegative, then if  $\hat{x}_0 \in R^n$  is a point for which there exists an exponent  $k_0$  such that  $\hat{x}_{k_0}$  is nonnegative, then all subsequent points in the trajectory emanating from  $\hat{x}_0$  will remain nonnegative. One result that we completed proving during the course of this grant represents a considerable improvement over results which we obtained previously in [N48]. It is the following:

**Theorem 2** ([N54]) *Let  $A$  be an  $n \times n$  essentially nonnegative matrix and consider the linear differential system (3.1). Let*

$$h(A) := \sup\{h > 0 \mid I + hA \geq 0\}. \quad (3.4)$$

*Then  $x_0 \in X_A(R_+^n)$  if and only if for any  $0 < h < h(A)$  there exists an index  $k_0$  such that the discrete trajectory of points (3.3) generated from  $\hat{x}_0 = x_0$  satisfies that*

$$\hat{x}_k \geq 0, \quad \forall k \geq k_0.$$

Notice that  $h(A)$  can be very large, it is  $+\infty$  if  $A$  is nonnegative, but in any case it depends only in the size of the diagonal entries of  $A$  and is not “infinitesimal”. What the result means is this: “**regardless of the extent to which the continuous and discrete trajectories diverge from each other, one becomes nonnegative if and only if the other one**

**does, provided only that the time-step  $h$  satisfies  $0 < h < h(A)$ .** This is a qualitative as well as a numerical statement about the behavior of the solutions to systems of ordinary differential equations whose coefficient matrix is essentially nonnegative. The proof of the above theorem is quite involved and is the subject of the manuscript "Reachability cones of essentially nonnegative matrices" which has just been accepted for publication in the journal of Linear and Multilinear Algebra. We mention that quite an important tool which was used in the proof of the theorem is taken from an earlier paper [N52] in which we considered an analytic approach to the question of existence of a nonnegative basis for the eigenspace of a nonnegative matrix corresponding to its Perron root.

The characterization of symbiosis points is done in [N59]. Decompose a point  $v \in X_A(R_+^n)$  into

$$v = v_+ - v_-,$$

where  $v_+$  is the join of all eigenspaces of  $A$  corresponding to eigenvalues with a nonnegative real part and  $v_-$  is in the join of all eigenspaces of  $A$  corresponding to eigenvalues with a negative real part. Thus  $v_-$  is in the stability part of the space since  $\lim_{t \rightarrow \infty} e^{tA} v_- = 0$ . For  $v$  we define the **invariant set of components of  $v$**  as the set

$$\mathcal{I}(v) = \{1 \leq i \leq n : (e^{tA} v)_i = (v_+)_i, \forall t \geq 0\}. \quad (3.5)$$

Thus  $\mathcal{I}(v)$  consists of the indices of the components of the vector  $v_+$  which remain invariant throughout the entire trajectory emanating from  $v_+$ . We prove the following characterization:

**Theorem 3** ([N59]) *Let  $A$  be an  $n \times n$  essentially nonnegative matrix. Then a vector  $v \in X_A(R_+^n)$  is a symbiosis point for (3.1) if and only if there exists a sufficiently large time  $t_0$  such that*

$$j \in \mathcal{I}(v) \implies 0 \leq (e^{tA} v_-)_j \downarrow 0, \quad \forall t \geq t_0, \quad (3.6)$$

where  $\mathcal{I}(v)$  is given in (3.5). Furthermore, if  $v \in X_A(R_+^n)$  is a symbiosis point, then for any  $j \in \mathcal{I}(v)$ ,  $(v_+)_j = 0$  if and only if  $(e^{tA} v)_j = 0$  for all  $t \geq 0$ .

We have two comments. First, in the spirit of Theorem 2 we can also characterize a symbiosis point  $v \in X_A(R_+^n)$  in terms of the nondecreasingness of finite differences sequences generated from  $v$  (similar to the way

in which the sequence in (3.3) is generated from  $x_0$ ) where the step size  $h$  satisfies (3.4). Second in the case when  $A$  is **weakly stable**, meaning its eigenvalue with the largest real part is the origin, then symbiosis points admit a matrix-combinatorial structure in the sense that it is possible to determine apriori which indices  $1 \leq i \leq n$  lie in  $\mathcal{I}(v)$  from a certain block directed graph of the matrix  $A$ . Both of these issues are addressed in [N59].

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